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# CALOGERO-MOSER SPACE, REDUCED RATIONAL CHEREDNIK ALGEBRAS AND TWO-SIDED CELLS.

I.G. GORDON AND M. MARTINO

ABSTRACT. We conjecture that the “nilpotent points” of Calogero-Moser space for reflection groups are parametrised naturally by the two-sided cells of the group with unequal parameters. The nilpotent points correspond to blocks of restricted Cherednik algebras and we describe these blocks in the case  $G = \mu_\ell \wr \mathfrak{S}_n$  and show that in type  $B$  our description produces an existing conjectural description of two-sided cells.

## 1. INTRODUCTION

1.1. Smooth points are all alike; every singular point is singular in its own way. Calogero-Moser space associated to the symmetric group has remarkable applications in a broad range of topics; in [3], Etingof and Ginzburg introduced a generalisation associated to any complex reflection group which has also found a variety of uses. The Calogero-Moser spaces associated to a complex reflection group, however, exhibit new behaviour: they are often singular. The nature of these singularities remains a mystery, but their existence has been used to solve the problem of the existence of crepant resolutions of symplectic quotient singularities. The generalised Calogero-Moser spaces are moduli spaces of representations of rational Cherednik algebras and so their geometry reflects the representation theory of these algebras: smooth points correspond to irreducible representations of maximal dimension; singular points to smaller, more interesting representations. In this note we conjecture a strong link between the representations corresponding to some particularly interesting “nilpotent points” of Calogero-Moser space and Kazhdan-Lusztig cell theory for Hecke algebras with unequal parameters. To justify the conjecture we give a combinatorial parametrisation of these points, thus answering a question of [7], and then relate this parametrisation to the conjectures of [1] on the cell theory for Weyl groups of type  $B$ .

1.2. Let  $W$  be a complex reflection group and  $\mathfrak{h}$  its reflection representation over  $\mathbb{C}$ . Let  $\mathcal{S}$  denote the set of complex reflections in  $W$ . Let  $\omega$  be the canonical symplectic form on  $V = \mathfrak{h} \oplus \mathfrak{h}^*$ . For  $s \in \mathcal{S}$ , let  $\omega_s$  be the skew-symmetric form that coincides with  $\omega$  on  $\text{im}(\text{id}_V - s)$  and has  $\ker(\text{id}_V - s)$  as its radical. Let  $\mathbf{c} : \mathcal{S} \rightarrow \mathbb{C}$  be a  $W$ -invariant function sending  $s$  to  $c_s$ . The *rational Cherednik algebra* at parameter  $t = 0$  (depending on  $\mathbf{c}$ ) is the quotient of the skew group algebra of the tensor algebra on  $V$ ,  $TV * W$ , by the relations

$$(1) \quad [x, y] = \sum_{s \in \mathcal{S}} c_s \omega_s(x, y) s$$

for all  $x, y \in V$ . This algebra is denoted by  $H_{\mathbf{c}}$ .

Let  $Z_{\mathbf{c}}$  denote the centre of  $H_{\mathbf{c}}$  and set  $A = \mathbb{C}[\mathfrak{h}^*]^W \otimes \mathbb{C}[\mathfrak{h}]^W$ . Thanks to [3, Proposition 4.15]  $A \subset Z_{\mathbf{c}}$  for any parameter  $\mathbf{c}$  and  $Z_{\mathbf{c}}$  is a free  $A$ -module of rank  $|W|$ . Let  $X_{\mathbf{c}}$  denote the spectrum of  $Z_{\mathbf{c}}$ : this is called the *Calogero-Moser space* associated to  $W$ . Corresponding to the inclusion  $A \subset Z_{\mathbf{c}}$  there is a finite surjective morphism  $\Upsilon_{\mathbf{c}} : X_{\mathbf{c}} \longrightarrow \mathfrak{h}^*/W \times \mathfrak{h}/W$ .

Let  $\mathfrak{m}$  be the homogeneous maximal ideal of  $A$ . The *restricted rational Cherednik algebra* is  $\overline{H}_{\mathbf{c}} = H_{\mathbf{c}}/\mathfrak{m}H_{\mathbf{c}}$ . By [3, PBW theorem 1.3] it has dimension  $|W|^3$  over  $\mathbb{C}$ . General theory asserts that the blocks of  $\overline{H}_{\mathbf{c}}$  are labelled by the closed points of the scheme-theoretic fibre  $\Upsilon_{\mathbf{c}}^*(0)$ . We call these points the *nilpotent points* of  $X_{\mathbf{c}}$ . By [7, 5.4] there is a surjective mapping

$$\Theta_{\mathbf{c}} : \text{Irr } W \longrightarrow \{\text{closed points of } \Upsilon_{\mathbf{c}}^*(0)\} = \{\text{blocks of } \overline{H}_{\mathbf{c}}\},$$

constructed by associating to any  $\lambda \in \text{Irr } W$  an indecomposable  $\overline{H}_{\mathbf{c}}$ -module, the baby Verma module  $M_{\mathbf{c}}(\lambda)$ . The fibres of  $\Theta_{\mathbf{c}}$  partition  $\text{Irr } W$ . We will call this the  $CM_{\mathbf{c}}$ -partition.

1.3. Let  $W$  be a Weyl group. Let  $\mathbf{L} : W \rightarrow \mathbb{Q}$  be a weight function (in the sense of [1, Section 2]). Let  $\mathcal{H}$  be the corresponding Iwahori-Hecke algebra at unequal parameters, an algebra over the group algebra of  $\mathbb{Q}$ ,  $A = \bigoplus_{q \in \mathbb{Q}} \mathbb{Z}v^q$ , which has a basis  $T_w$  for  $w \in W$ , with multiplication given by the rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) = l(w) + 1 \\ T_{sw} + (v^{\mathbf{L}(s)} - v^{-\mathbf{L}(s)})T_w & \text{if } l(sw) = l(w) - 1 \end{cases}$$

where  $s \in \mathcal{S}$  and  $w \in W$ . There is an associated partition of  $W$  into two-sided cells, see [9, Chapter 8]. We call these the  $KL_{\mathbf{L}}$ -cells.

**Conjecture.** Let  $W$  be a Weyl group and let  $\mathbf{L}$  be the weight function generated by  $\mathbf{L}(s) = c_s$  for each  $s \in \mathcal{S}$ .

- (1) There is a natural identification of the  $CM_{\mathbf{c}}$ -partition and the  $KL_{\mathbf{L}}$ -partition; this is induced by attaching a  $KL_{\mathbf{L}}$ -cell to an irreducible  $W$ -representation via the asymptotic algebra  $J$ , [9, 20.2].
- (2) Let  $\mathcal{F}$  be a  $KL_{\mathbf{L}}$ -cell of  $W$  and let  $M_{\mathcal{F}}$  be the closed point of  $\Upsilon_{\mathbf{c}}^*(0)$  corresponding to  $\mathcal{F}$  by (1). Then  $\dim_{\mathbb{C}}(\Upsilon_{\mathbf{c}}^*(0)_{M_{\mathcal{F}}}) = |\mathcal{F}|$ .

The existence of the asymptotic algebra mentioned in (1) is still a conjecture, depending on Lusztig's conjectures P1-P15 in [9, Conjecture 14.2].

1.4. This conjecture generalises the known results about the blocks of  $\overline{H}_{\mathbf{c}}$  and about the fibre  $\Upsilon_{\mathbf{c}}^*(0)$ .

- [7, Corollary 5.8] If  $X_{\mathbf{c}}$  is smooth then  $\Theta_{\mathbf{c}}$  is bijective, making the  $CM_{\mathbf{c}}$ -partition trivial. If  $S \in \text{Irr } W$  then  $\dim_{\mathbb{C}}(\Upsilon_{\mathbf{c}}^*(0)_{M_S}) = \dim_{\mathbb{C}}(S)^2$ .
- $\Theta_{\mathbf{c}}$  is not bijective when  $W$  is a finite Coxeter group of type  $D_{2n}$ ,  $E$ ,  $F$ ,  $H$  or  $I_2(m)$  ( $m \geq 5$ ), [7, Proposition 7.3]. In all of these cases there are non-trivial two-sided cells.

- Both the  $a$ -function and the  $c$ -function are constant across fibres of  $\Theta_{\mathbf{c}}$ , [6, Lemma 5.3 and Proposition 9.2]. This should be a property of two-sided cells.

An advantage of the Cherednik algebras is that the  $CM_{\mathbf{c}}$ -partition exists for any complex reflection group whereas, at the moment, a cell theory only exists for Coxeter groups.

1.5. In Theorem 2.5 we will give a combinatorial description of the  $CM_{\mathbf{c}}$ -partition when  $W = G(\ell, 1, n) = \mu_{\ell} \wr \mathfrak{S}_n$ , and then in Theorem 3.3 we will provide evidence for the conjecture by showing that the  $CM_{\mathbf{c}}$ -partition agrees with the conjectural description of the  $KL_{\mathbf{L}}$ -partition for  $W = G(2, 1, n)$ , the Weyl group of type  $B_n$ , given in [1, Section 4.2].

## 2. BLOCKS FOR $W = G(\ell, 1, n)$

2.1. Let  $\ell$  and  $n$  be positive integers. Let  $\mu_{\ell}$  be the group of  $\ell$ -th roots of unity in  $\mathbb{C}$  with generator  $\sigma$  and let  $\mathfrak{S}_n$  be the symmetric group on  $n$  letters. Let  $W$  be the wreath product  $G(\ell, 1, n) = \mu_{\ell} \wr \mathfrak{S}_n = (\mu_{\ell})^n \rtimes \mathfrak{S}_n$  acting naturally on  $\mathfrak{h} = \mathbb{C}^n$ .

2.2. Let  $\mathcal{P}(n)$  denote the set of partitions of  $n$  and  $\mathcal{P}(\ell, n)$  the set of  $\ell$ -multipartitions of  $n$ . The set  $\text{Irr } W$  can be identified naturally with  $\mathcal{P}(\ell, n)$  so that the trivial representation corresponds to  $((n), \emptyset, \dots, \emptyset)$ , e.g. [8, Theorem 4.4.3]. Given an element  $\mathbf{s} \in \mathbb{Z}_0^{\ell} = \{(s_1, \dots, s_{\ell}) \in \mathbb{Z}^{\ell} : s_1 + \dots + s_{\ell} = 0\}$  there is an associated  $\ell$ -core (a partition from which no  $\ell$ -hooks can be removed). The inverse of the process assigning to a partition its  $\ell$ -core and  $\ell$ -quotient defines a bijection

$$(2) \quad \mathbb{Z}_0^{\ell} \times \coprod_n \mathcal{P}(\ell, n) \longrightarrow \coprod_n \mathcal{P}(n), \quad (\mathbf{s}, \boldsymbol{\lambda}) \mapsto \tau_{\mathbf{s}}(\boldsymbol{\lambda}).$$

A detailed discussion of this can be found in [8, Section 2.7] or [6, Section 6].

2.3. The Young diagram of a partition  $\lambda$  will always be justified to the northwest (one of the authors is English); we will label the box in the  $p$ th row and  $q$ th column of  $\lambda$  by  $s_{pq}$ . With this convention the residue of  $s_{pq}$  is defined to be congruence class of  $p - q$  modulo  $\ell$ . Recall that  $s_{pq}$  is said to be  $j$ -removable for some  $0 \leq j \leq \ell - 1$  if it has residue  $j$  and if  $\lambda \setminus \{s_{pq}\}$  is the Young diagram of another partition, a predecessor of  $\lambda$ . We say that  $s_{pq}$  is  $j$ -addable to  $\lambda \setminus \{s_{pq}\}$ .

Let  $J \subseteq \{0, \dots, \ell - 1\}$ . We define the  $J$ -heart of  $\lambda$  to be the sub-partition of  $\lambda$  which is obtained by removing as often as possible  $j$ -removable boxes with  $j \in J$  from  $\lambda$  and its predecessors. A subset of  $\mathcal{P}(n)$  whose elements are the partitions with a given  $J$ -heart is called a  $J$ -class.

2.4. We will use the “stability parameters” of [6]  $\boldsymbol{\theta}(\mathbf{c}) = (\theta_0, \dots, \theta_{\ell-1})$  defined by  $\theta_k = -\delta_{0k}c_{(i,j)} + \sum_{t=1}^{\ell-1} \eta^{tk} c_{\sigma^t}$  for  $0 \leq k \leq \ell - 1$ ,  $\eta$  a primitive  $\ell$ -th root of unity and an arbitrary transposition  $(i, j) \in \mathfrak{S}_n$ : they contain the same information as  $\mathbf{c}$ . Following [6, Theorem 4.1] we set  $\Theta = \{(\theta_0, \dots, \theta_{\ell-1}) \in \mathbb{Q}^{\ell}\}$  and  $\Theta_1 = \{\boldsymbol{\theta} \in \Theta : \theta_0 + \dots + \theta_{\ell-1} = 1\}$ .

Let  $\tilde{\mathfrak{S}}_\ell$  denote the affine symmetric group with generators  $\{\sigma_0, \dots, \sigma_{\ell-1}\}$ . It acts naturally on  $\Theta$  by  $\sigma_j \cdot (\theta_0, \dots, \theta_{\ell-1}) = (\theta_0, \dots, \theta_{j-1} + \theta_j, -\theta_j, \theta_j + \theta_{j+1}, \dots, \theta_{\ell-1})$ . This restricts to the affine reflection representation on  $\Theta_1$ : the *walls* of  $\Theta_1$  are the reflecting hyperplanes and the *alcoves* of  $\Theta_1$  are the connected components of (the real extension of)  $\Theta_1 \setminus \{\text{walls}\}$ . Let  $A_0$  be the alcove containing the point  $\ell^{-1}(1, \dots, 1)$ : its closure is a fundamental domain for the action of  $\tilde{\mathfrak{S}}_\ell$  on  $\Theta_1$ . The stabiliser of a point  $\theta \in \overline{A_0}$  is a standard parabolic subgroup of  $\tilde{\mathfrak{S}}_\ell$  generated by simple reflections  $\{\sigma_j : j \in J\}$  for some subset  $J \subseteq \{0, \dots, \ell-1\}$ . We call this subset the *type* of  $\theta$ . The type of an arbitrary point  $\theta \in \Theta_1$  is defined to be the type of its conjugate in  $\overline{A_0}$ .

2.5. We have an isomorphism  $\tilde{\mathfrak{S}}_\ell \cong \mathbb{Z}_0^\ell \times \mathfrak{S}_\ell$  with  $\mathfrak{S}_\ell = \langle \sigma_1, \dots, \sigma_{\ell-1} \rangle$  and the elements of  $\mathbb{Z}_0^\ell$  corresponding to translations by elements of the dual root lattice  $\mathbb{Z}R^\vee$ . The symmetric group  $\mathfrak{S}_\ell$  acts on  $\mathcal{P}(\ell, n)$  by permuting the partitions comprising an  $\ell$ -multipartition.

**Theorem.** Assume that  $\theta(\mathbf{c}) \in \Theta_1$ , so that  $\theta(\mathbf{c})$  had type  $J$  and belongs to  $(\mathbf{s}, w) \cdot \overline{A_0}$  for some  $(\mathbf{s}, w) \in \tilde{\mathfrak{S}}_\ell$ . Then  $\lambda, \mu \in \text{Irr} W = \mathcal{P}(\ell, n)$  belong to the same block of  $\overline{H}_{\mathbf{c}}$  if and only if  $\tau_{\mathbf{s}}(w \cdot \lambda)$  and  $\tau_{\mathbf{s}}(w \cdot \mu)$  belong to the same  $J$ -class. In other words, the  $CM_{\mathbf{c}}$ -partition is governed by the  $J$ -classes.

*Proof.* Rescaling gives an isomorphism between  $\overline{H}_{\mathbf{c}}$  and  $\overline{H}_{\mathbf{c}/2}$  so we can replace  $\mathbf{c}$  by  $\mathbf{c}/2$ . By [7, 5.4] we must show that the baby Verma modules  $M_{\mathbf{c}/2}(\lambda)$  and  $M_{\mathbf{c}/2}(\mu)$  give rise to the same closed point of  $\Upsilon_{\mathbf{c}/2}^*(0)$  if and only if  $\tau_{\mathbf{s}}(w \cdot \lambda)$  and  $\tau_{\mathbf{s}}(w \cdot \mu)$  have the same  $J$ -class. But the closed points of  $\Upsilon_{\mathbf{c}/2}^*(0)$  correspond to the  $\mathbb{C}^*$ -fixed points of  $X_{\mathbf{c}/2}$  under the action induced from the grading on  $H_{\mathbf{c}/2}$  which assigns degree 1, respectively  $-1$  and  $0$ , to non-zero elements of  $\mathfrak{h}$ , respectively  $\mathfrak{h}^*$  and  $W$ . By [6, Theorem 3.10] these agree with the  $\mathbb{C}^*$ -fixed points on the affine quiver variety  $\mathcal{X}_{\theta(\mathbf{c})}(n)$  and hence, thanks to [6, Equation (3)] to the  $\mathbb{C}^*$ -fixed points on the Nakajima quiver variety  $\mathcal{M}_{\theta(\mathbf{c})}(n)$ . Now the result follows since the combinatorial description of these fixed points in [6, Proposition 8.3(i)] is exactly the one in the statement of the theorem.  $\square$

2.6. **Remarks.** (1) The assumption  $\theta(\mathbf{c}) \in \Theta_1$  imposes two restrictions. First it places a rationality condition on the entries of  $\mathbf{c}$ ; guided by corresponding results for Hecke algebras, [2, Theorem 1.1], we hope that this is not really a serious restriction. Second it forces  $c_{(i,j)} \neq 0$ ; if  $c_{(i,j)} \neq 0$  then we can rescale to produce an isomorphism  $H_{\mathbf{c}} \cong H_{\lambda_{\mathbf{c}}}$  and hence ensure  $\theta_0 + \dots + \theta_{\ell-1} = 1$ .

(2) A generic choice of  $\theta(\mathbf{c}) \in \Theta_1$  will have type  $J = \emptyset$ . The corresponding  $CM_{\mathbf{c}}$ -partition will then be trivial and thus  $X_{\mathbf{c}}$  will be smooth, [3, Corollary 1.14(i)].

### 3. THE CASE $W = G(2, 1, n)$

3.1. We now focus on the situation where  $W = G(2, 1, n)$ , the Weyl group of type  $B_n$ . Here there are two conjugacy classes of reflections  $s$  and  $t$ , containing  $(i, j)$  and  $\sigma$  respectively. We will always assume that  $\mathbf{c} = (c_s, c_t) \in \mathbb{Q}^2$  has the property that  $c_s, c_t \neq 0$ . Corresponding to the two group homomorphisms  $\epsilon_1, \epsilon_2 : W \rightarrow \mathbb{C}^*$ ,  $\epsilon_k(i, j) = (-1)^k$  for all  $(i, j) \in s$  and  $\epsilon_k(\sigma) = (-1)^{k+1}$ , there exist algebra isomorphisms

$H_{(c_s, c_t)} \cong H_{(-c_s, c_t)}$  and  $H_{(c_s, c_t)} \cong H_{(c_s, -c_t)}$ , [5, 5.4.1]. So, without loss of generality, we may assume that  $\mathbf{c} \in \mathbb{Q}_{>0}^2$ .

3.2. There is a conjectural description of the two-sided cells in [1, Section 4.2] which we recall very briefly; more details can be found in both [loc.cit] and [10].

We assume  $\mathbf{L}(s) = a, \mathbf{L}(t) = b$  with  $a, b \in \mathbb{Q}_{>0}$  and set  $d = b/a$ . If  $d \notin \mathbb{Z}$  then the partition is conjectured to be trivial, [1, Conjecture A(c)]. If  $d = r + 1 \in \mathbb{Z}$  then let  $\mathcal{P}_r(n)$  be the set of partitions of size  $\frac{1}{2}r(r+1) + 2n$  with 2-core  $(r, r-1, \dots, 1)$ . A domino tableau  $T$  on  $\lambda \in \mathcal{P}_r(n)$  is a filling of the Young diagram of  $\lambda$  with 0's in the 2-core and then  $n$  dominoes in the remaining boxes, each labelled by a distinct integer between 1 and  $n$  which are weakly increasing both vertically and horizontally. There is a process called *moving through an open cycle* which leads to an equivalence relation on the set of domino tableaux. This in turn leads to an equivalence relation on partitions in  $\mathcal{P}_r(n)$  where  $\lambda$  and  $\mu$  are related if there is a sequence of partitions  $\lambda = \lambda_0, \lambda_1, \dots, \lambda_{s-1}, \lambda_s = \mu$  such that for each  $1 \leq i \leq s$ ,  $\lambda_{i-1}$  and  $\lambda_i$  are the underlying shapes of some domino tableaux related by moving through an open cycle. The equivalence classes of this relation are called  $r$ -cells. [1, Conjecture D] conjectures that the two-sided cells are in natural bijection with the  $r$ -cells.

3.3. The result of this section is the following.

**Theorem.** *Under the bijection (2) the  $CM_{\mathbf{c}}$ -partition of  $\text{Irr} W$  is identified with the above conjectural description of the  $KL_{\mathbf{L}}$ -partition for  $\mathbf{L}(s) = c_s, \mathbf{L}(t) = c_t$ .*

This theorem shows that core-quotient algorithm provides a natural identification of the  $CM_{\mathbf{c}}$ -partition and the conjectural  $KL_{\mathbf{L}}$ -partition. We do not know in general whether Lusztig's conjectured mapping from  $\text{Irr} W$  to the  $KL_{\mathbf{L}}$ -cells is given by this algorithm.

There are special cases where [1, Conjecture D] has been checked – for instance the asymptotic case  $c_t > (n-1)c_s$ , [1, Remark 1.3] – and thus in those cases we really do get a natural identification between the  $CM_{\mathbf{c}}$ -classes and  $KL_{\mathbf{c}}$ -cells.

3.4. We will need the following technical lemma to prove the theorem.

**Lemma.** *Let  $\lambda \in \mathcal{P}_r(n)$  and set  $j = r$  modulo 2 with  $j \in \{0, 1\}$ . Suppose that  $s_{pq}$  is a  $j$ -removable box and  $s_{tu}$  is a  $j$ -addable box such that  $p \geq t$  and  $q \leq u$  and there are no other  $j$ -addable or  $j$ -removable boxes,  $s_{vw}$ , with  $p \geq v \geq t$  and  $q \leq w \leq u$ . Then there is a domino tableau  $T$  of shape  $\lambda$  and an open cycle  $c$  of  $T$  such that the shape of the domino obtained by moving through  $c$  is obtained by replacing  $s_{pq}$  with  $s_{tu}$ .*

*Proof.* We use the notation of [10, Sections 2.1 and 2.3] freely. We consider the rim ribbon which begins at  $s_{pq}$  and ends at  $s_{t, u-1}$ . We claim that this rim ribbon can be paved by dominoes. In fact this is a general property of a ribbon connecting a box,  $s$ , of residue  $j$  and a box,  $e$ , of residue  $j+1$ . Let  $R$  be such a ribbon. If  $R$  contains only two boxes then  $R = \{s, e\}$  so it is clear. In general the box adjacent to  $s$ , say  $s_{\text{ad}}$ , has residue  $j+1$  so that  $R \setminus \{s, s_{\text{ad}}\}$  is a ribbon of smaller length and so the result follows by induction. In our situation

we can specify more. Starting at  $s_{pq}$  we tile our rim ribbon,  $R$ , as far as possible with vertical dominoes up to and including  $D = \{s_{p-m+1,q}, s_{p-m,q}\}$  where  $s_{p-m,q}$  has residue  $j+1$ . If  $s_{p-m,q} = s_{t,u-1}$  we have finished our tiling. Otherwise  $s_{p-m,q+1} \in R$  so the square  $s_{p-m,q+1}$  will be  $j$ -removable unless  $\{s_{p-m,q+1}, s_{p-m,q+2}\} \subseteq R$ . We now tile with as many horizontal dominoes as possible until we get to  $E = \{s_{p-m,q+k-1}, s_{p-m,q+k}\}$  with  $s_{p-m,q+k}$  having residue  $j+1$ . If  $s_{p-m,q+k} = s_{t,u-1}$  then our tiling stops. Otherwise we must have the next domino as  $F = \{s_{p-m-1,q+k}, s_{p-m-2,q+k}\}$  to avoid  $s_{p-m,q+k+1}$  being  $j$ -addable. We can now repeat this process to obtain our tiling of  $R$ . From this description we obtain the following consequences. Let  $s_{vw} \in R$  have residue  $j+1$  and suppose that  $s_{vw} \neq s_{t,u-1}$ . Then

- (i) The domino which contains  $s_{vw}$  is either of the form  $\{s_{v+1,w}, s_{vw}\}$  or  $\{s_{v,w-1}, s_{vw}\}$ ;
- (ii) If  $s_{v-1,w+1} \in \lambda$  then  $s_{v,w+1} \in R$ . Furthermore, if  $s_{vw}$  is contained in a horizontal domino then  $s_{v-1,w+1} \notin R$ ;
- (iii) If  $s_{v-1,w+1} \notin \lambda$  then  $s_{v-1,w} \in R$ .

Let  $R$  denote the rim ribbon above and suppose it can be tiled by  $t$  dominoes. Let  $\mu$  be the shape  $\lambda \setminus R$ . In particular  $\mu$  contains  $\frac{1}{2}r(r+1) + 2(n-t)$  squares. By the previous paragraph and [8, Lemma 2.7.13],  $\mu$  is a Young diagram with the same 2-core as  $\lambda$  and so there exists a  $T' \in \mathcal{P}_r(n-t)$  with shape  $\mu$ . Take such a  $T'$  filled with the numbers 1 to  $n-t$ . Now add  $R$  to  $T'$ . We can tile  $R$  by dominoes by the previous paragraph and we fill the dominoes with the numbers  $n-t+1, \dots, n$  where the filling is weakly increasing on the rows and columns of  $R$ . This gives a domino tableau  $T = T' \cup R$  of shape  $\lambda$ .

We claim that  $R \subseteq T$  is an open cycle, and that when we move through this cycle we remove  $s_{pq}$  from  $T$  and add  $s_{tu}$ . This will prove the lemma.

As we have seen in (i) a domino  $D \subseteq R$  is either of the form  $D = \{s_{vw}, s_{v+1,w}\}$  or  $D = \{s_{v,w-1}, s_{vw}\}$  with  $s_{vw}$  having residue  $j+1$ . In the case that  $D = \{s_{vw}, s_{v+1,w}\}$  we have to study the square  $s_{v-1,w+1}$  to calculate  $D'$ . One of two things can happen. If this box does not belong to  $T$  then  $D' = \{s_{vw}, s_{v-1,w}\}$  and  $D' \subseteq R$  by (iii). Otherwise  $s_{v-1,w+1}$  does belong to  $T$ . In this situation the box is not in the rim so is filled with a lower value than  $D$  and so  $D' = \{s_{vw}, s_{v,w+1}\}$ . In particular, by (ii) above either  $D' \subseteq R$  or  $D' = \{s_{t,u-1}, s_{tu}\}$ .

Now suppose  $D = \{s_{v,w-1}, s_{vw}\}$ . If the square  $s_{v-1,w+1}$  is not in  $T$  then  $D' = \{s_{v-1,w}, s_{vw}\}$  and  $D' \subseteq R$  by (iii). If  $s_{v-1,w+1}$  is in  $T$  then it is not in the rim by (ii) and so is filled with a value lower than that of  $D$ . Thus  $D' = \{s_{vw}, s_{v,w+1}\}$  and  $D' \subseteq R$  unless  $s_{vw} = s_{t,u-1}$ , in which case  $D' = \{s_{t,u-1}, s_{tu}\}$ .

It is now clear that  $R$  is a cycle and moving through this cycle changes the shape of  $\lambda$  by removing  $s_{pq}$  and adding  $s_{tu}$ .  $\square$

**3.5. Proof of Theorem 3.3.** We have  $\theta(\mathbf{c}) = (-c_s + c_t, -c_t)$  and by rescaling, see Remark 2.6(1), we consider  $\theta'(\mathbf{c}) = (1 - \frac{c_t}{c_s}, \frac{c_t}{c_s}) \in \Theta_1$ . The action of  $\tilde{\mathfrak{S}}_2$  on  $\Theta_1$  is given by  $\sigma_0 \cdot (\theta_0, \theta_1) = (-\theta_0, \theta_1 + 2\theta_0)$  and  $\sigma_1 \cdot (\theta_0, \theta_1) = (\theta_0 + 2\theta_1, -\theta_1)$ . The walls are  $\{(d, -d+1) \in \Theta_1 : d \in \mathbb{Z}\}$ ; they are of type  $\{0\}$  if  $d \in 2\mathbb{Z}$  and of type  $\{1\}$  if  $d \in 1 + 2\mathbb{Z}$ . The fundamental alcove is  $A_0 = \{(d, -d+1) : 0 < d < 1\}$ ,

and the alcove  $A_r = \{(d, -d + 1) : r < d < r + 1\}$  is then labelled by either  $((\frac{r}{2}, \frac{-r}{2}), e) \in \mathbb{Z}_0^2 \times \mathfrak{S}_2$  or  $((\frac{-r+1}{2}, \frac{r-1}{2}), \sigma_1) \in \mathbb{Z}_0^2 \times \mathfrak{S}_2$  depending on whether  $r$  is even or odd.

If  $\frac{c_t}{c_s} \notin \mathbb{Z}$  then  $\theta'(\mathbf{c})$  has type  $\emptyset$  and the  $CM_{\mathbf{c}}$ -partition of  $\text{lrr } W$  is trivial by Theorem 2.5 since  $\emptyset$ -classes are all singletons: this agrees with the conjectured triviality of the two-sided cells in this case. Thus we may assume that  $r = \frac{c_t}{c_s} - 1 \in \mathbb{Z}_{\geq 0}$ . Then  $\theta'(\mathbf{c}) = (-r, r + 1)$  will be in the closure of two alcoves,  $A_{-r-1}$  and  $A_{-r}$ . We consider the latter. Let  $\mathbf{s}$  be the element in  $\mathbb{Z}_0^2$  coming from the labelling of  $A_{-r}$ ; then  $\tau_{\mathbf{s}}$  of (2) produces a bijection between  $\text{lrr } W = \mathcal{P}(2, n)$  and  $\mathcal{P}_r(n)$ . If we set  $j = r$  modulo 2 and  $J = \{j\}$ , the content of the theorem is simply the assertion that the  $r$ -cells in  $\mathcal{P}_r(n)$  consist of the partitions in  $\mathcal{P}_r(n)$  with the same  $J$ -heart.

Let us show first that if  $\lambda, \mu \in \mathcal{P}_r(n)$  have the same  $J$ -heart, say  $\rho$ , then they belong to the same  $r$ -cell. The  $J$ -heart has no  $j$ -removable boxes, but we can construct the partition  $\mu$  from  $\rho$  by adding, say,  $t$   $j$ -addable boxes. Now let  $\nu$  be the partition obtained from  $\rho$  by adding  $t$   $j$ -addable boxes as far left as possible. We note that by [8, Theorem 2.7.41]  $\nu \in \mathcal{P}_r(n)$ . Of course  $\mu$  and  $\nu$  could be the same, but usually they will be different. Now we apply Lemma 3.4 again and again to  $\nu$ , taking first the rightmost  $j$ -removable box from  $\nu$  to the position of the rightmost  $j$ -removable box on  $\mu$  and then repeating with the next  $j$ -removable box on the successor of  $\nu$ . We continue until we have obtained a partition with shape  $\mu$ . By Lemma 3.4, this process is obtained by moving through open cycles. On the other hand, we can perform this operation in the opposite direction to move from  $\lambda$  to  $\nu$  via open cycles (for this we use the same algorithm and the fact that for a cycle  $c$ , moving through  $c$  twice takes us back where we started, [4, Proof of Proposition 1.5.31]). It follows that  $\lambda$  and  $\mu$  belong to the same  $r$ -cell.

Finally, we need to see that if  $\lambda, \mu \in \mathcal{P}_r(n)$  belong to the same  $r$ -cell, then they have the same  $J$ -heart. For this it is enough to assume that  $\mu$  is the shape of a tableau obtained by moving through an open cycle on a tableau of shape  $\lambda$ . But in this case the underlying shapes differ only in some  $j$ -removable boxes, [10, Section 2.3] and so they necessarily have the same  $J$ -heart.  $\square$

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